Generalized Fourier Series for Solutions of Linear Differential Equations

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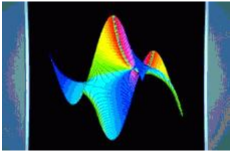

I Introduction

Generalized Fourier Series

$$f(x) = \sum a_n \psi_n(x)$$

Some Examples

$$\sin(x) = 2\sum_{n=0}^{\infty} (-1)^n \mathrm{J}_{2n}(x)$$

$$\arccos(x) = \frac{1}{2\pi} T_0(x) - \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2 \pi} T_{2n+1}(x)$$

$$\operatorname{erf}(x) = 2\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{1}{\sqrt{\pi}\,(2n+1)\,n!}\, {}_1F_1\left(\left.\begin{matrix} n+\frac{1}{2} \\ 2n+2 \end{matrix}\right| -x\right)$$

More generally $(\psi_n(x))_{n\in\mathbb{N}}$ can be an orthogonal basis of a Hilbert space.

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More generally $(\psi_n(x))_{n\in\mathbb{N}}$ can be an orthogonal basis of a Hilbert space.
Applications: Good approximation properties.

Our framework

Families of functions $\psi_n(x)$ with two special properties

Mult by x (P_x)

$$\mathrm{Rec}_{x2}\left(x\psi_n(x)\right) = \mathrm{Rec}_{x1}\left(\psi_n(x)\right)$$

Example

- Monomial polynomials ($M_n = x^n$)
- All orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions

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- Monomial polynomials ($M_n = x^n$) — $xM_n = M_{n+1}$
- All orthogonal polynomials — $2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$
- Bessel functions — $\frac{1}{n}\left(x\mathrm{J}_{n+1} - x\mathrm{J}_{n-1}\right) = 2\mathrm{J}_n$
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$$M_n' = nM_{n-1}$$

$$\frac{1}{n+1}T_{n+1}'(x) - \frac{1}{n-1}T_{n-1}'(x) = 2T_n(x)$$

$$2\mathrm{J}_n'(x) = \mathrm{J}_{n-1}(x) - \mathrm{J}_{n+1}(x)$$

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This is our data-structure for $\psi_n(x)$

Main Idea

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If $\psi_n(x)$ satisfies (P_x) and (P_∂), for any $f(x) = \sum a_n \psi_n(x)$ solution of a linear differential equation with polynomial coefficients, the coefficients a_n are cancelled by a linear recurrence relation with polynomial coefficients.

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Application:

- Efficient numerical computation of the coefficients.
- Computation of closed-form for the coefficients (when possible).

Previous work

- Clenshaw (1957): numerical scheme to compute the coefficients when $\psi_n(x) = T_n(x)$ (Chebyshev series).
- Lewanowicz (1976-2004): algorithms to compute a recurrence relation when ψ_n is an orthogonal or semi-orthogonal polynomials family.
- Rebillard and Zakrajšek (2006): General algorithm computing a recurrence relation when ψ_n is a family of hypergeometric polynomials
- Benoit and Salvy (2009) : Simple unified presentation and complexity analysis of the previous algorithms using Fractions of recurrence relations when $\psi_n = T_n$. New and fast algorithm to compute the Chebyshev recurrence.

New Results (2010)

- Simple unified presentation of the previous algorithms using Pairs of recurrence relations.
- New general algorithm computing the recurrence relation of the coefficients for a Generalized Fourier Series when $\psi_n(x)$ satisfies (P_x) and (P_∂).

II Pairs of Recurrence Relations

Examples: Chebyshev case $(f(x) = \sum u_n T_n(x))$

Basic rules:

$$xf = \sum a_n T_n \quad \xrightarrow{(P_x)} \quad a_n = \frac{u_{n-1} + u_{n+1}}{2}$$

$$f' = \sum b_n T_n \quad \xrightarrow{(P_\partial)} \quad b_{n-1} - b_{n+1} = 2n u_n.$$

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Combine:

$$f' + 2xf = \sum c_n T_n \quad \xrightarrow{(P_\partial + 2P_x)} \quad c_{n-1} - c_{n+1} = \mathrm{Rec}_1(u_n).$$

Application: Chebyshev series for $\exp(-x^2)$.

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$$(f' + 2xf)' = \sum d_n T_n \quad \xrightarrow{(P_\partial)} \quad d_{n-1} - d_{n+1} = 2n c_n,$$

$$\rightarrow \quad \mathrm{Rec}_2(d_n) = \mathrm{Rec}_3(u_n),$$

$$(f' + 2xf)' - 2f = \sum e_n T_n \quad \rightarrow \quad \mathrm{Rec}_4(e_n) = \mathrm{Rec}_5(u_n).$$

Application: Chebyshev series for $\mathrm{erf}(x)$.

Rings of Pairs of Recurrence Relations

Theorem (Least Common Left Multiple (Ore 33))

Given Rec_1 *and* Rec_2*, there exists a recurrence relation* Rec *and a pair* $\left(\widetilde{\mathrm{Rec}_1}, \widetilde{\mathrm{Rec}_2}\right)$ *such that for all sequences* $(u_n)_{n\in\mathbb{N}}$ *:*

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Operation 1: Addition

$\mathrm{Rec}_1(a_n) = \mathrm{Rec}_3(u_n)$, $\mathrm{Rec}_2(b_n) = \mathrm{Rec}_4(u_n)$

$\rightarrow \mathrm{Rec}(a_n + b_n) = \left(\widetilde{\mathrm{Rec}_1} \circ \mathrm{Rec}_3 + \widetilde{\mathrm{Rec}_2} \circ \mathrm{Rec}_4\right)(u_n)$.

Operation 2: Composition

$\mathrm{Rec}_1(a_n) = \mathrm{Rec}_3(u_n)$, $\mathrm{Rec}_4(b_n) = \mathrm{Rec}_2(a_n)$

$\rightarrow \widetilde{\mathrm{Rec}_1} \circ \mathrm{Rec}_2(u_n) = \widetilde{\mathrm{Rec}_2} \circ \mathrm{Rec}_3(b_n)$.

Main Result

Main Result : Morphism

There exists a morphism such that if $f = \sum u_n \psi_n(x)$ and $g = \sum v_n \psi_n(x)$ are related by $L(f) = g$ (L a linear differential operator), then:

$$\varphi(L) = (\mathrm{Rec}_1, \mathrm{Rec}_2) \quad \text{with} \quad \mathrm{Rec}_1(u_n) = \mathrm{Rec}_2(v_n)$$

In particular if $L(f) = 0$, then $Rec_1(u_n) = 0$.

Definition of the Morphism φ

$$f = \sum u_n \psi_n(x) \quad g = \sum v_n \psi_n(x)$$

$$\mathrm{Rec}_{x2}\left(x\psi_n(x)\right) = \mathrm{Rec}_{x1}\left(\psi_n(x)\right) \xrightarrow{\varphi(x)} \text{if } xf = g, \text{ then } \overline{\mathrm{Rec}}_{x2}\left(u_n\right) = \overline{\mathrm{Rec}}_{x1}\left(v_n\right)$$

$$\mathrm{Rec}_{\partial 2}\left(\psi_n'(x)\right) = \mathrm{Rec}_{\partial 1}\left(\psi_n(x)\right) \xrightarrow{\varphi(\partial)} \text{if } f' = g, \text{ then } \overline{\mathrm{Rec}}_{\partial 1}\left(u_n\right) = \overline{\mathrm{Rec}}_{\partial 2}\left(v_n\right)$$

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Example for Chebyshev series:

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$$
$$\frac{T_{n+1}'(x)}{n+1} - \frac{T_{n-1}'(x)}{n-1} = 2T_n(x)$$

$\xrightarrow{\varphi}$

$$u_{n+1} + u_{n-1} = 2v_n$$
$$2u_n = \frac{1}{n}\left(v_{n-1} - v_{n+1}\right)$$

Example for Bessel series

$$\frac{1}{n}\left(x\mathrm{J}_{n+1} - x\mathrm{J}_{n-1}\right) = 2\mathrm{J}_n$$
$$2\mathrm{J}_n'(x) = \mathrm{J}_{n-1}(x) - \mathrm{J}_{n+1}(x)$$

$\xrightarrow{\varphi}$

$$2u_n = \frac{v_{n+1}}{n+1} + \frac{v_{n-1}}{n-1}$$
$$u_{n+1} - u_{n-1} = 2v_n$$

General Algorithm

Recall

- Definition of $\varphi(x)$ and $\varphi(\partial)$
- Algorithm to compute addition and composition between two pairs

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General Algorithm

We deduce from this morphism a general Horner-like algorithm to compute the recurrence relation satisfy by the coefficients of a generalized Fourier series solution of a linear differential equation.

III Recurrences of Smaller Order

Greatest Common Left Divisor and Reduction of Order

GCLD

Given a pair $(\mathrm{Rec}_1, \mathrm{Rec}_2)$, the Euclidean algorithm computes the greatest recurrence relation Rec (GCLD) such that there exists a pair $\left(\widetilde{\mathrm{Rec}_1}, \widetilde{\mathrm{Rec}_2}\right)$ with the following relations for all sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$:

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Aim:

Find cases where for all coefficients of Generalized Fourier series (u_n, v_n), we have :

$$\mathrm{Rec}_1\,(u_n) = \mathrm{Rec}_2\,(v_n) \Longrightarrow \widetilde{\mathrm{Rec}_1}\,(u_n) = \widetilde{\mathrm{Rec}_2}\,(v_n)$$

GCD for reduction of order

Theorem

Given L a linear differential operator, $f = \sum u_n\psi_n(x)$, $g = \sum v_n\psi_n(x)$ such that $L(f) = g$ and a pair $(\mathrm{Rec}_1, \mathrm{Rec}_2) = \varphi(L)$. We have

$$\widetilde{\mathrm{Rec}_1}\,(u_n) = \widetilde{\mathrm{Rec}_2}\,(v_n)$$

if ψ_n is any of:

- *Classical orthogonal polynomials*
- *Bessel functions*
- *Hypergeometric polynomials*

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Application: Adaptation of the previous algorithm

At the end of the previous algorithm, add a final step:
Remove the *GCLD* of the two recurrence relations of the pair.

Example of reduction for Chebyshev series

$$\sqrt{1-x^2} = \sum_{n\in\mathbb{N}} \frac{4}{\pi(2n+1)} T_{2n}(x) = \sum_{n\in\mathbb{N}} c_n T_n(x)$$

$\sqrt{1-x^2}$ is the solution of the differential equation:

$$xy(x) + (1-x^2)y'(x) = 0$$

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With the general algorithm we obtain the pair of recurrence relations :

$\mathrm{Rec}_1\,(u_n) = (n+3)u_{n+2} - 2nu_n + (n-3)u_{n-2}$ and $\mathrm{Rec}_2\,(v_n) = 2\,(-v_{n+1} + v_{n-1})\,.$

We deduce : $(n+3)c_{n+2} - 2nc_n + (n-3)c_{n-2} = 0.$

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$\widetilde{\mathrm{Rec}_1}(u_n) = (n+2)u_{n+1} - (n-2)u_{n-1}$ and $\widetilde{\mathrm{Rec}_2}(v_n) = -2v_{n+1} + 2v_{n-1}$.

We deduce : $(n+2)c_{n+1} - (n-2)c_{n-1} = 0.$

IV Conclusion

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Contributions:

- Use of Pairs of recurrence relations.
- New general algorithm.
- Use of the GCLD to reduce order of the recurrence.

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Perspectives:

- Computation of the recurrence of minimal order.
- Numerical computation of the coefficients.
- Closed form for the coefficients.

Example

$$\operatorname{erf}(x) = \sum_{n=0}^{\infty} 2\,\frac{4^{-n}\,(-1)^{n}\,{}_1F_1(n+1/2;\,2\,n+2;\,-1)}{\sqrt{\pi}\,(2\,n+1)\,n!}\,T_{2\,n+1}(x)\,.$$